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Open membranes, p-branes and noncommutativity of boundary string coordinates

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Abstract: We study the dynamics of an open membrane with a cylindrical topology, in the background of a constant three form, whose boundary is attached to p-branes. The boundary closed string is coupled to a two form potential to ensure gauge invariance. We use the action, due to Bergshoeff, London and Townsend, to study the noncommutativity properties of the boundary string coordinates. The constrained hamiltonian formalism due to Dirac is used to derive the noncommutativity of coordinates. The chain of constraints is found to be finite for a suitable gauge choice, unlike the case of the static gauge, where the chain has an infinite sequence of terms. It is conjectured that the formulation of closed string field theory may necessitate introduction of a star product which is both non commutative and non associative.

Keywords: M-Theory, p-branes, Non-Commutative Geometry.

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Contents

1. Introduction

Recently, the study of non-commutative geometry, from the perspective of string theory, has attracted considerable attention. The noncommutativity of the target space coordinates becomes manifest when a constant background NS two form potential is introduced along the D-brane [1]. In the presence of the two-form potential, the end points of the open strings attached to the D-brane do not commute. The intimate connection between string theory, non-commutative geometry and non-commutative Yang-Mills theories has been investigated from diverse points of view by Seiberg and Witten in their seminal paper [2]. It is natural, therefore, to examine the corresponding issue when an open membrane ends on a D-brane under an analogous situation. In this case, one would envisage a D-brane in flat space in the background of a three-form tensor potential, with components tangential to brane coordinates, since a membrane couples to the third rank antisymmetric tensor just as a string couples to the two-form potential.

The motivation to study this problem arises from the conjectures that membranes provide a clue to understanding M-theory. The five perturbatively consistent string theories are believed to be different phases of the underlying M-theory and it is argued that the low energy limit of M-theory is described by the $D = 11$ supergravity theory, just as low energy limits of various super string theories go over to the corresponding supergravity theories in ten dimensions. It is also well known that membrane and five brane appear naturally in eleven-dimensional supergravity and already, there have been attempts to study the M5-brane in the background of a constant 3-form [2]. Another motivation to study properties of open membranes arises from the perspective of OM theory [3]. It has been conjectured that five-dimensional non-commutative open string theory on an M5-brane, in the strong coupling limit, decouples from gravity, for a critical value of the field strength, and becomes equivalent to a theory of light open membranes (OM theory decouples from gravity with a near critical 3-form field strength.). Furthermore, some progress has been made in examining how noncommutativity may arise by using open membranes as probes on branes [4, 5].

It is worth while to mention, at this point, that the study of noncommutativity properties of open membranes ending on branes involves some subtleties compared to the case where one considers an open string ending on D-branes. The action, in the case of string theory, is chosen to be of Polyakov type, whereas for membranes the conventional choice has been the Nambu-Goto action, with additional terms which are introduced from various other considerations. Furthermore, the equations of motion for a Nambu-Goto action are non linear, in contrast to the linear equations arising in the case of the Polyakov formulation of the string theory. Therefore, one has to resort to some approximation scheme while investigating the noncommutativity aspects of such a theory. We mention *en passant* that one of the easiest ways to bring out the noncommutativity for an open string-D-brane system, in the presence of a NS B-field, is to scale the metric and the B-field suitably and then take $\alpha' \to 0$ limit. However, in the case of a membrane-brane system, there is no such obvious limit, as has been emphasized by several authors [4, 5].

In this paper, we have investigated the noncommutativity property of an open membrane-brane configuration from a different perspective. In the first place, in order to circumvent some of the technical difficulties encountered in the Nambu-Goto action, we adopt a modified version of the action due to Bergshoeff, London and Townsend [6], where one introduces a two-form gauge potential in the world volume and where the tension is given the status of a dynamical variable (depends on world volume coordinates). The equations of motion derived from the BLT action coincide with those coming from the Nambu-Goto action, when one substitutes the solutions of the auxiliary equations for the world volume two-form potential as well as the tension, into the equation of motion for the membrane coordinates. In fact, one can write down an additional contribution due to the space-time dependent three form background coupled to the membrane and study the symmetries associated with such a system [7]. As we shall show, there are certain advantages in adopting the BLT action. When an open string end is attached to the D-brane, one introduces the coupling of the open string to a gauge field background and the gauge invariance of the world sheet action implies that the field strength associated with the gauge potential be constant for constant B-field. Note that in the analogous situation, where the open membrane (more details about the membrane configuration later, which we basically choose to have a cylindrical topology) ends on the brane, the boundary is that of a closed string and, therefore, a gauge symmetry demands that one needs to introduce a two-form B-field coupled to the boundary string just as a boundary D0-brane couples to a gauge potential. Thus, we are set out to study noncommutativity properties of the ends of the membrane which are coordinates of a string, depending on world volume time and one of its spatial coordinates.

The constrained hamiltonian formalism due to Dirac [8] is one of the most elegant and powerful techniques in which one can exhibit this feature of the theory. Let us recapitulate how noncommutativity arises in an open string theory when a constant v-field is introduced along the brane direction. In the absence of B, the coordinates along the Dp-brane satisfy Neumann boundary condition, $\partial_\sigma X^\mu = 0, \mu = 0, 1, \ldots, p$, at the boundary $\sigma = 0, \pi$; here we take the target space metric to be the flat Minkowski metric, $\eta_{\mu\nu}$. When B is introduced, the boundary condition changes to a mixed condition, namely, $\eta_{\mu\nu}\partial_\sigma X^\nu + B_{\mu\nu}\partial_\tau X^\nu = 0$ at the boundaries. It has been suggested that the boundary condition can be used as a primary constraint after eliminating $\dot{X}^\mu$ in favor of the canonical momenta of the string coordinates. One can then use the procedure of Dirac to derive all the secondary constraints of the theory, identify the second class constraints and finally evaluate the Dirac brackets between various phase space variables [9] to see noncommutativity of coordinates.

It is important to note that supersymmetry is not an essential ingredient in the study of noncommutativity in the geometry of a brane in a constant anti-symmetric field background. Therefore, we shall consider a bosonic open membrane ending on a brane. However, it is also important to keep in mind that this is a simplified model and within the context of M-theory, it will be essential to consider super membranes. In that frame work, one should consider the action in the background of the massless fields of eleven-dimensional supergravity such as the graviton and the 3-form tensor. The gauge invariance will be lost if we do not introduce the 2-form B-field for the open membrane, as alluded to above. When one introduces open super membranes, all the supersymmetries are broken, even in the flat Minkowski space. However, in the presence of topological defects as backgrounds [10], it is possible to construct supersymmetric actions for an open supermembrane. These defects have a natural interpretation analogous to the end of the world 9-plane in the Horava-Witten construction [11]. Thus, keeping in mind that one is likely to deal with supermembrane theories, we consider an open membrane ending on a p-brane with constant target space metric and three form potential along with a 2-form B-field coupled to the closed string on the boundary.

Our approach is similar to that of Kawamoto and Sasakura. However, with the modified form of the action, we are able to make some head way with the computation of the matrix of constraints as well as the evaluation of the Dirac brackets (DB) in

JHEP04(2001)016

a systematic manner, without linearizing the action as was done at the outset in [5]. It is worthwhile to point out one interesting feature that arises in the computation of the secondary constraints for the problem under consideration. To start with, in the Dirac formalism, one identifies the primary constraints and requires that the canonical Poisson bracket (PB) of these constraints with the total hamiltonian must vanish. In other words, the constraints must be stationary under time evolution. As a consequence, either one generates new constraints, which in turn generate further constraints and this goes on *ad infinitum*, or that this process terminates in the sense that one does not generate any new constraint after a finite number of iterations. However, as we will show, in the case of the membrane, unlike in string theories, the velocities, $\dot{X}^\mu$ cannot be written in terms of the canonical momenta and, consequently, the boundary conditions cannot be written as primary constraints on the phase space, *unless one chooses a gauge to begin with.* In ref [5], the Dirac analysis was carried out in the static gauge (in addition to the linearized approximation for the action) and an infinite chain of constraints was obtained. In fact, a similar situation also arises in the case of open strings ending on D-branes. Namely, one knows that incorporation of boundary conditions as primary constraints, in general, leads to an infinite chain of constraints. Surprisingly, however, we find that with an alternate, suitable choice of gauge (one is free to choose a gauge), the constraint chain for the same membrane system terminates. In other words, in this alternate gauge, after a finite number of iterations, new constraints are not generated. On the other hand, as a consistency check, when we do resort to the static gauge, we find an infinite chain of constraints, similar to those of ref [5]; the two chains do not coincide since the starting point of hamiltonian analysis are different (namely, the starting actions are different), due to different gauge choices. It is well known that noncommutativity depends on the choice of gauge (which, however, cannot be eliminated). That the nature of the chain of constraints also depends on the choice of gauge is something that we had not seen earlier.

The paper is organized as follows. In section 2, we first consider the Nambu-Goto action for the membrane and discuss some of its salient features. The resulting equations of motion are presented along with the relevant boundary conditions. Subsequently, the alternate action due to Bergshoeff, London and Townsend [6] is introduced; it is recalled how the original equations of motion are recovered from this action. In section 3, we proceed systematically with the hamiltonian analysis to identify the primary constraints. We show how a gauge choice becomes essential to carry out the Dirac procedure and choose a suitable gauge in order to facilitate the analysis of constraints. In section 4, we carry out the analysis of constraints in some detail and show that, in this particular choice of gauge, the chain of constraints terminates. In section 5, the evaluation of the Dirac brackets is discussed, where we explicitly determine the Dirac brackets to linear order in the background antisymmetric field and demonstrate how noncommutativity of the coordinates on the boundary arises. We summarize our results in section 6 and end with conclusions. In appendix A, we briefly indicate how the static gauge leads to an infinite chain of constraints and in appendix B, we point out the essential structure of the Dirac brackets to quadratic order in the anti-symmetric background field.

2. The action

It is well known that the action for a membrane can be described by a Nambu-Goto action, much like the action for a string. Thus, for example, the world volume action for an open membrane in 11-dimensions, interacting with an anti-symmetric background field, C_{MNP} can be described by the action

$$S = T\left(\int_{\Sigma_3} d^3\xi \left(\sqrt{g} - \frac{1}{6}\varepsilon^{ijk}\partial_i X^M \partial_j X^N \partial_k X^P C_{MNP}\right) + \int_{\partial\Sigma_3} B\right), \tag{2.1}$$

where T represents the tension of the membrane. Here $M, N, P = 0, 1, \ldots, 10$ are indices of the eleven-dimensional target space, $\xi = (\tau, \sigma_1, \sigma_2)$ are the coordinates of the world volume of the membrane with the corresponding indices taking values $i, j, k = 0, 1, 2$, $g = \det g_{ij}$, where $g_{ij} = G_{MN}\partial_i X^M \partial_j X^N$; is the induced metric on the membrane (We use a metric with signature $(+, -, -)$ on the world volume and one with signature $(+, -, -, \ldots, -)$ in the target space.). In addition to the anti-symmetric tensor background on the world volume, here we also have a boundary term which is required for gauge invariance of the action. Namely, with the boundary term, the action is invariant under gauge transformations of the kind

$$\begin{aligned} C &\longrightarrow C + d\Lambda\,, \\ B &\longrightarrow B - \Lambda\,, \end{aligned} \tag{2.2}$$

where Λ is the local, two form parameter of transformation. Combining C and dB into a single 3-form

$$A = C + dB \tag{2.3}$$

we can rewrite the action (2.1) in the following form:

$$S = T\int_{\Sigma_3} d^3\xi \left(\sqrt{g} - \frac{1}{6}\varepsilon^{ijk}\partial_i X^M \partial_j X^N \partial_k X^P A_{MNP}\right). \tag{2.4}$$

For simplicity, we are going to choose both G_{MN} and A_{MNP} to be constants (In fact, we will choose $G_{MN} = \eta_{MN}$ from now on.). In such a case, it is well known that this action can describe an open membrane ending on p-branes in the eleven-dimensional target space.

In fact, let us note that the action (2.4) leads to the equations of motion

$$\partial_i\left(\sqrt{g}g^{ij}\eta_{MN}\partial_j X^N - \frac{1}{2}\varepsilon^{ijk}A_{MNP}\partial_j X^N \partial_k X^P\right) = 0 \tag{2.5}$$

along with the boundary conditions

$$n_i\left(\sqrt{g}g^{ij}\eta_{MN}\partial_j X^N - \frac{1}{2}\varepsilon^{ijk}A_{MNP}\partial_j X^N\partial_k X^P\right)\delta X^M = 0\,, \tag{2.6}$$

where n_i represents the unit normal vector. If we assume that our membrane has a cylindrical topology characterized by $0 \leq \sigma_1 \leq \pi$, $0 \leq \sigma_2 \leq 2\pi$ in units of some length scale, this can, in fact, describe an open membrane terminating on p-branes. The hamiltonian analysis, following from the action (2.4), has been carried out in [5] with the gauge choices (for reparameterization invariance)

$$\begin{aligned} X^0 &= \tau\,, & \tau &\in (\infty,\infty)\,, \\ X^9 &= \sigma_1 L\,, & \sigma_1 &\in [0,\pi]\,, \\ X^{10} &= \sigma_2 R\,, & \sigma_2 &\in [0,2\pi)\,, \end{aligned} \tag{2.7}$$

where the radius of the compactified direction, X^{10}, is R. (As we will see later, in the case of membranes, gauge fixing becomes essential before carrying out an analysis of constraints.) It has been shown, in such a case, how noncommutativity arises in the Dirac brackets. However, because of the non-linear nature of such a formulation, the analysis in [5] was done only in the linearized approximation (as well as other approximations) and it would be nice to see if one can do the analysis without restricting to the linearized approximation.

An alternate description for the membrane is through the first order action of the form (due to Bergshoeff, London and Townsend) [6]

$$S_{\rm BLT} = \int_{\Sigma_3} d^3\xi\,\frac{1}{2V}(g - \widetilde{\mathcal{F}}^2)\,. \tag{2.8}$$

Here we have defined

$$\widetilde{\mathcal{F}} \equiv \varepsilon^{ijk}\widetilde{\mathcal{F}}_{ijk} = \varepsilon^{ijk}\left(F_{ijk} + \frac{1}{6}A_{ijk}\right), \tag{2.9}$$

where

$$\begin{aligned} F_{ijk} &= \partial_{[i}U_{jk]} = \partial_i U_{jk} + \text{cyclic}\,, \\ A_{ijk} &= \partial_i X^M\partial_j X^N\partial_k X^P A_{MNP} \end{aligned} \tag{2.10}$$

with A_{MNP} defined earlier. Clearly, $V(\xi)$ and $U_{ij}(\xi)$ are auxiliary field variables. If we now look at the equations of motion following from the action (2.8), we obtain

$$\partial_i\left(\frac{1}{V}gg^{ij}\partial_j X^N\eta_{MN} - \frac{1}{2}\frac{\widetilde{\mathcal{F}}}{V}\varepsilon^{ijk}\partial_j X^N\partial_k X^P A_{MNP}\right) = 0\,, \tag{2.11}$$

$$\partial_i\left(\frac{\widetilde{\mathcal{F}}}{V}\right) = 0\,, \tag{2.12}$$

$$g = \widetilde{\mathcal{F}}^2\,. \tag{2.13}$$

JHEP04(2001)016

The constraint equations in (2.12) and (2.13) can be easily solved. Choosing the solutions as

$$\widetilde{\mathcal{F}} = TV = \sqrt{g} \tag{2.14}$$

with T representing the tension in eq. (2.4) and substituting these into eq. (2.11), we see that the dynamical equation for the coordinates coincides with the one in (2.5) following from the action in (2.4).

Similarly, we note that the boundary conditions following from the action (2.8) take the forms

$$n_i \left(\left(\frac{1}{V}\right) g g^{ij} \partial_j X^N \eta_{MN} - \left(\frac{\widetilde{\mathcal{F}}}{2V}\right) \varepsilon^{ijk} \partial_j X^N \partial_k X^P A_{MNP} \right) \delta X^M = 0\,, \tag{2.15}$$

$$\varepsilon^{ijk} n_i \delta U_{jk} = 0\,. \tag{2.16}$$

Using the solutions (2.14) for the auxiliary field equations, we see that eq. (2.15) reproduces the boundary condition (2.6) following from the action (2.4). The second boundary condition, (2.16), can simply be satisfied by choosing U_{01} to be a constant along σ_1.

Due to the cylindrical topology of the membrane, in the presence of p-branes, the boundary condition (2.6) or equivalently (2.15) reduces to

$$\sqrt{g} g^{1j} \partial_j X_\mu - \frac{1}{2} \varepsilon^{1jk} \partial_j X^\nu \partial_k X^\rho A_{\mu\nu\rho} \,|_{\sigma_1 = 0,\pi} = 0\,, \qquad \mu, \nu = 0, 1, \ldots, p\,, \tag{2.17}$$

$$X^a = x_0^a\,, \qquad a = p+1, \ldots, 10\,, \tag{2.18}$$

where the coordinates, x_0^a, specify the positions of the p-branes at the two boundaries $\sigma_1 = 0, \pi$.

3. Gauge fixing and the hamiltonian

We take the first order action (2.8) as the starting point of our canonical description of the system. The goal, of course, is to find the hamiltonian, implement the boundary conditions as primary constraints, determine all the constraints of the theory and derive the Dirac brackets for the system. The problem, however, is that the action for the membrane is highly constrained. Of course, even string theory has constraints. However, as we will show next, the constraints in the case of membranes are much more difficult to deal with and necessitate gauge fixing.

To see the difficulty, let us note that the lagrangian density of the action (2.8) is singular in the sense that the determinant of the coefficient matrix multiplying the quadratic terms in velocities (namely, the Hessian matrix)

$$\frac{\partial^2 L_{\rm BLT}}{\partial \dot{X}^M \partial \dot{X}^N}\,,$$

where

$$L_{\mathrm{BLT}} = \frac{1}{2V}(g - \widetilde{\mathcal{F}}^2) \tag{3.1}$$

vanishes. This, of course, reflects the fact that there are constraints in the system. To see the nature of the constraints in a simple manner, let us introduce the following notations. Let us separate the world volume indices into time and space as $i = (0, a)$ where $a = 1, 2$. In that case, it is easy to check that the determinant of the metric can be written in the factorized form

$$g = \det g_{ij} = \left(g_{00} - g_{0a}\overline{g}^{ab} g_{b0}\right) \det g_{ab}\,, \tag{3.2}$$

where, as we have defined earlier,

$$g_{ab} = \partial_a X^M \partial_b X^N \eta_{MN}\,, \qquad a, b = 1, 2$$

and $\overline{g}^{ab} (\neq g^{ab})$ is the inverse of this in the two-dimensional subspace. With this notation, it is clear that the momentum, conjugate to X^M, is given by

$$\begin{aligned} P_M &= \frac{\partial L_{\mathrm{BLT}}}{\partial \dot{X}^M} \\ &= \frac{1}{V}\left(\left(\dot{X}_M - \partial_a X_M \overline{g}^{ab} g_{b0}\right) \det g_{ab} - \frac{1}{2}\widetilde{\mathcal{F}}\varepsilon^{ab}\partial_a X^N \partial_b X^P A_{MNP}\right) \end{aligned}$$

so that we can write

$$\left(\eta_{MN} - \partial_a X_M \overline{g}^{ab} \partial_b X_N\right) \dot{X}^N = \frac{1}{\overline{g}}\left(V P_M + \frac{1}{2}\widetilde{\mathcal{F}}\varepsilon^{ab}\partial_a X^N \partial_b X^P A_{MNP}\right), \tag{3.3}$$

where we have introduced the notation $\overline{g} = \det g_{ab}$. There are several things to note from the structure of eq. (3.3). First, it follows immediately from this equation as well as the definitions of g_{ab}, that

$$P_M \partial_a X^M = 0\,, \qquad a = 1, 2\,. \tag{3.4}$$

These are, in fact, the two generators of reparameterization symmetry along the σ_1, σ_2 directions. The analogous relation in string theory is

$$P_M \partial_\sigma X^M = 0\,.$$

where$P_M \partial_\sigma X^M$ corresponds to the generator of σ reparameterization invariance. The generator of reparameterization along the τ direction is the hamiltonian as we will see later. However, more important from our point of view is the fact that the matrix multiplying $\dot{X}^N$, on the l.h.s. of eq. (3.3), is a transverse projection operator and, therefore, the velocities cannot be expressed in terms of the phase space variables. Since the boundary conditions in eq. (2.17) involve velocities, this also means that, as it stands, the boundary conditions cannot be written as primary conditions on the phase space so that the Dirac analysis cannot be carried out in a conventional manner. We would like to emphasize that this is a new feature that is not present in the hamiltonian analysis of strings. It is not hard to see that the origin of this difficulty lies in the reparameterization invariance in our theory. If we fix a gauge, then the velocities can, in fact, be expressed in terms of phase space variables and the Dirac analysis can be carried out. It is well known that local symmetries, such as reparameterization invariance, lead to first class constraints which need to be gauge fixed. However, normally, this is done after the constraint analysis has been completed and all the constraints have been classified into first class and second class constraints. In the present case, on the other hand, we cannot even carry out the constraint analysis, because the boundary conditions cannot be written as phase space constraints, *unless we fix a gauge.*

It is clear, therefore, that to carry out the hamiltonian analysis for the membrane, it is necessary to start with a gauge fixed action instead of the action in (2.8). A conventional gauge choice (static gauge) such as

$$X^0 = \tau\,, \qquad g_{0a} = 0\,, \qquad a = 1, 2$$

would fix all the reparameterization invariances of the theory and would allow the velocities to be inverted. In this case, one can carry out the constraint analysis, which leads to the expected behavior that the boundary conditions induce an infinite chain of constraints (We will describe this briefly in appendix A.). However, the infinite number of such constraints are highly non linear and, therefore, are not readily amenable to calculating Dirac brackets.

We will, therefore, choose an alternate gauge condition which brings out an interesting feature of our analysis, namely, that with a suitable gauge choice, the chain of constraints can terminate. A finite set of constraints is clearly much easier to handle, in calculating Dirac brackets. Let us look at the action (2.8) in the gauge

$$g_{0a} = 0\,, \qquad a = 1, 2\,. \tag{3.5}$$

Namely, we are going to fix only the reparameterization invariance along the spatial directions (This is equivalent to fixing two of the spatial coordinates which we take to be normal to the brane.).

In this gauge, action (2.8) takes the form

$$S = \int_{\Sigma_3} d^3\xi\, \frac{1}{2V}\left(\overline{g}\dot{X}^M\dot{X}_M - \widetilde{\mathcal{F}}^2\right). \tag{3.6}$$

This action still has a τ reparameterization invariance, but, as we will see, it does not interfere with the hamiltonian analysis of the system. It is now straightforward to determine the canonical momenta from eq. (3.6) and they take the forms

$$
\begin{aligned}
P_M &= \frac{\partial L}{\partial \dot{X}^M} = \frac{\overline{g}}{V}\dot{X}_M - \frac{\widetilde{\mathcal{F}}}{2V}\varepsilon^{0ab}\partial_a X^N \partial_b X^P A_{MNP}\,, \\
\Pi^{(U)ab} &= \frac{\partial L}{\partial \dot{U}_{ab}} = -\frac{3\widetilde{\mathcal{F}}}{V}\varepsilon^{oab} = -\frac{3\widetilde{\mathcal{F}}}{V}\varepsilon^{ab}\,, \\
\Pi^{(U)oa} &= \frac{\partial L}{\partial \dot{U}_{0a}} \approx 0\,, \\
P_V &= \frac{\partial L}{\partial \dot{V}} \approx 0\,,
\end{aligned}
\tag{3.7}
$$

Here $P_M, \Pi^{(U)ab}, \Pi^{(U)oa}$, and P_V are momenta conjugate to the fields X^M, U_{ab}, U_{0a} and V, respectively and it is clear that we have two primary constraints in the theory given by the last two equations in (3.7).

It is also clear from eq. (3.7) that the velocities can be inverted and we have

$$
\dot{X}_M = \frac{V}{\overline{g}}\mathcal{P}_M\,, \tag{3.8}
$$

where we have defined

$$
\mathcal{P}_M \equiv P_M - \frac{1}{6}\Pi^{(U)ab}\partial_a X^N \partial_b X^P A_{MNP}\,. \tag{3.9}
$$

The canonical hamiltonian can now be determined and has the form

$$
H_{\text{can}} = P^M \dot{X}_M + \Pi^{(U)ab}\dot{U}_{ab} - L \tag{3.10}
$$

and, using (3.7) and (3.8), can be written in the form

$$
H_{\text{can}} = \frac{V}{2\overline{g}}\mathcal{P}^2 - \frac{V}{72}\left(\Pi^{(U)ab}\varepsilon_{ab}\right)^2 + 2\Pi^{(U)ab}\partial_a U_{0b}\,. \tag{3.11}
$$

We can now derive the boundary conditions following from the action (3.6) which, using eq. (3.8), can be converted to phase space constraints of the form

$$
\left(\overline{g}^{a1}\partial_a X_\mu \mathcal{P}^2 + \frac{1}{3}\Pi^{(U)a1}\mathcal{P}^\nu \partial_a X^\lambda A_{\mu\nu\lambda}\right)\Big|_{\sigma_1=0,\pi} \approx 0\,. \tag{3.12}
$$

With this, we are now ready to carry out the constraint analysis for the system, which we take up in the next section.

We note here that in the constraint analysis of ref [5], the three form background was chosen to be purely magnetic, namely, $A_{0IJ} = 0, I, J = 1, 2, \ldots, p$ and only the spatial components, A_{IJK}, are non zero. However, our constraint analysis in section 4 and the evaluation of the Dirac brackets in section 5 do not crucially depend on such a choice for the A-field. On the other hand, let us note that if A_{0IJ} were non zero, then it is likely that the DB of the time component of the boundary string coordinate $X^0(\tau,\sigma_2)$ with a spatial component of string coordinate will be non zero and the same result will hold for the DB of time-time coordinates. Although, the consequences of noncommutativity of these coordinates have not been investigated in detail, it is well known from the analysis of the corresponding D-brane open string that such noncommutativity of space-time and time-time coordinates pose some difficulty in formulating the perturbative Feynman rules [12]. On the other hand, there are persuasive reasons to believe that space and time coordinates may not commute, based on some general arguments as well as certain string and M-theoretic analysis leading to space-time uncertainty relations [13, 14, 15]. In view of all these possibilities, we have kept our analysis quite general without choosing any specific form for the A-field .

4. Constraint analysis

In section 3, we determined the canonical hamiltonian of the system in the gauge (3.5) and noted the two primary constraints following from the definition of the conjugate momenta. Adding to these the boundary constraints, the complete set of primary constraints can be written as

$$\begin{aligned}
\varphi_1 &= P_V \approx 0\,,\\
\varphi_2^a &= \Pi^{(U)oa} \approx 0\,,\\
\varphi_{3\mu} &= \left(\overline{g}^{a1}\partial_a X_\mu \mathcal{P}^2 + \frac{1}{3}\Pi^{(U)a1}\mathcal{P}^\nu \partial_a X^\lambda A_{\mu\nu\lambda}\right)\delta(\sigma_1) \approx 0\,,\\
\varphi_{4\mu} &= \left(\overline{g}^{a1}\partial_a X_\mu \mathcal{P}^2 + \frac{1}{3}\Pi^{(U)a1}\mathcal{P}^\nu \partial_a X^\lambda A_{\mu\nu\lambda}\right)\delta(\sigma_1-\pi) \approx 0\,.
\end{aligned}\tag{4.1}$$

Adding these primary constraints to the canonical hamiltonian (3.11), we obtain the hamiltonian for the system to be

$$H = \frac{V}{2\overline{g}}\mathcal{P}^2 - \frac{V}{72}\left(\Pi^{(U)ab}\varepsilon_{ab}\right)^2 + 2\Pi^{(U)ab}\partial_a U_{0b} + c\varphi_1 + k_a\varphi_2^a + \lambda^\mu\varphi_{3\mu} + \widetilde{\lambda}^\mu\varphi_{4\mu}\,, \tag{4.2}$$

where c, k_a, λ^μ, and $\widetilde{\lambda}^\mu$ are Lagrange multipliers.

The analysis for the consistency of constraints can now be carried out in a straightforward manner. The consistency condition

$$\dot{\varphi}_1 = \left\{\varphi_1, \int H\right\} \approx 0 \tag{4.3}$$

leads to

$$\begin{aligned}
\lambda^\mu &= \widetilde{\lambda}^\mu = 0\,,\\
\varphi_5 &= \frac{\mathcal{P}^2}{2\overline{g}} - \frac{1}{72}\left(\Pi^{(U)ab}\varepsilon_{ab}\right)^2 \approx 0\,.
\end{aligned}\tag{4.4}$$

JHEP04(2001)016

There are a couple of points to be noted from this. First, the vanishing of the Lagrange multipliers, λ^μ and $\widetilde{\lambda}^\mu$, can be understood intuitively as follows. We have already seen in eq. (3.8) that the velocities can be expressed in terms of the phase space variables as

$$\dot{X}_M = \frac{V}{\overline{g}}\mathcal{P}_M \,.$$

On the other hand, if we calculate directly, using the hamiltonian (4.2), we have

$$\dot{X}_M = \left\{X_M, \int H\right\} \tag{4.5}$$

and we find that the two are compatible only if $\lambda^\mu = 0 = \widetilde{\lambda}^\mu$. The second point to note from eq. (4.4) is that the non evolution of the constraint φ_1 leads to a secondary constraint, φ_5, which, as we will see shortly, is simply the constraint reflecting the τ reparameterization invariance of the theory. But, for the present, we only note that the consistency of this secondary constraint leads to no further constraint.

Let us next note that the consistency of the second constraint in eq. (4.1),

$$\dot{\varphi}_2^a = \left\{\varphi_2^a, \int H\right\} \approx 0$$

leads to a secondary constraint of the form

$$\varphi_6^a = \partial_b \Pi^{(U)ba} \approx 0 \,. \tag{4.6}$$

This is the analog of Gauss' law in electrodynamics and it can be easily checked that the consistency of this constraint leads to no new constraints. Furthermore, it is clear now that with eq. (4.6), the constraint in eq. (4.4) simply says that the canonical hamiltonian (and, therefore, H) vanishes in this theory, which is a reflection of the τ reparameterization invariance of the theory. Notice that, since U_{ab} is a cyclic variable in our theory, it follows that $\Pi^{(U)ab}$ is a constant of motion. Thus, together with eq. (4.6), this implies that $\Pi^{(U)ab}$ is truly a constant, which in turn implies, from eq. (4.4), that $\mathcal{P}^2/\overline{g}$ is a constant. Without loss of generality we choose

$$\frac{\mathcal{P}^2}{\overline{g}} = 1 \,. \tag{4.7}$$

With the help of eq. (4.7), the boundary conditions can now be rewritten (reduced) as

$$\begin{aligned}
\varphi_{3\mu} &= \left(\overline{g}\,\overline{g}^{a1}\partial_a X_\mu + \frac{1}{3}\Pi^{(U)a1}\mathcal{P}^\nu \partial_a X^\lambda A_{\mu\nu\lambda}\right)\delta(\sigma_1) \approx 0 \,, \\
\varphi_{4\mu} &= \left(\overline{g}\,\overline{g}^{a1}\partial_a X_\mu + \frac{1}{3}\Pi^{(U)a1}\mathcal{P}^\nu \partial_a X^\lambda A_{\mu\nu\lambda}\right)\delta(\sigma_1 - \pi) \approx 0 \,.
\end{aligned} \tag{4.8}$$

Consistency of the boundary constraint $\varphi_{3\mu} \approx 0$,

$$\dot{\varphi}_{3\mu} = \left\{ \varphi_{3\mu}, \int H \right\} \approx 0$$

leads to the secondary constraint,

$$\varphi_{7\mu} = \delta(\sigma_1) \left[\bar{g}^{a1} \partial_a \left[\frac{V}{\bar{g}} \mathcal{P}_\mu \right] + \varepsilon^{ab} \partial_a X_\mu \partial_{(b} X^\lambda \partial_{2)} \left[\frac{V}{\bar{g}} \mathcal{P}_\lambda \right] + \frac{V}{3\bar{g}} \Pi^{(U)a1} A_{\mu\nu\lambda} \mathcal{P}^\nu \partial_a \mathcal{P}^\lambda - \right.$$
$$\left. - \frac{1}{3} \Pi^{(U)a1} A_{\mu\nu\lambda} \partial_a X^\lambda \partial_c [V \bar{g}^{bc} \partial_b X^\nu] \right]. \tag{4.9}$$

In deriving eq. (4.9), we have used the relations $\{\mathcal{P}_\mu(\sigma), \mathcal{P}_\nu(\sigma')\} \approx 0$, which follows from the Gauss' law constraint, as well as the following PB

$$\left\{ \mathcal{P}_\mu \mathcal{P}^\mu(\sigma), \frac{1}{\bar{g}(\sigma')} \right\} = -\frac{4}{\bar{g}(\sigma')} \bar{g}^{ab}(\sigma') \partial_a X_\nu(\sigma') \partial_b \delta\left(\sigma - \sigma^{'}\right). \tag{4.10}$$

Since the hamiltonian (4.2) contains a term of the form cP_V and the secondary constraint $\varphi_{7\mu}$ depends on V as well as ∂V, consistency of this new constraint

$$\dot{\varphi}_{7\mu} = \left\{ \varphi_{7\mu}, \int H \right\} \approx 0$$

simply determines the Lagrange multiplier c and leads to no further constraint. The expression for the Lagrange multiplier is complicated and its explicit form is not very crucial for our analysis; therefore, we do not present it here. An identical analysis goes through for the constraint $\varphi_{4\mu}$ at the other boundary and generates only one secondary constraint $\varphi_{8\mu}$, whose structure is identical to that of $\varphi_{7\mu}$ except that it is at the other boundary. What we have found is truly remarkable. We may recall that the boundary constraints, in the context of string theory as well as in the analysis of constraints in [5], led to an infinite chain of constraints. In contrast, we find that in a particular gauge, the boundary constraints for the case at hand lead to only one secondary constraint at each boundary. Namely, the chain of constraints actually terminates which is quite desirable from the point of view of calculating Dirac brackets. We have also carried out the constraint analysis, starting from our action in the static gauge and indeed, we find that the infinite chain of constraints do appear. That is, had we worked in a different gauge, there would be an infinite chain of constraints as normally expected. It is known in the literature that the noncommutativity arising in Dirac brackets depends on the gauge choice (which, however, cannot be eliminated). However, what we find here is that the nature of the constraint chain itself depends on the choice of gauge. This is, in fact, the only example that we are aware of, where the constraint chain for boundary conditions terminates [1]

[1]M.M. Sheikh-Jabbari and collaborators have also noted such a phenomema in a different model (unpublished), private communication.

JHEP04(2001)016

5. Dirac brackets

Since we have determined all the constraints of our theory, it is now straightforward, in principle, to determine the Dirac brackets. However, we note that the boundary constraints, (3.12) and (4.9), are, in particular, highly non linear and, consequently, evaluation of the inverse of the matrix of constraints is, in general, a very difficult problem. Things, however, do simplify enormously if we use a weak field approximation for $A_{\mu\nu\lambda}$. In this case, it is easy to determine the inverse of the matrix of constraints in an iterative manner to any order in the $A_{\mu\nu\lambda}$ field. Let us demonstrate this by first calculating the Dirac bracket to linear order in $A_{\mu\nu\lambda}$. In appendix B, we will indicate the structure of the Dirac brackets to second order in this field.

Let us note that among the entire set of constraints including the primary constraints $\varphi_1, \varphi_2^a, \varphi_{3\mu}, \varphi_{4\mu}$ and the secondary constraints $\varphi_5, \varphi_6^a, \varphi_{7\mu}, \varphi_{8\mu}$, only the boundary constraints $\varphi_{3\mu}, \varphi_{7\mu}, \varphi_{4\mu}$ and $\varphi_{8\mu}$ are second class constraints. The other constraints are all first class and can be handled by choosing appropriate gauge fixing conditions. These (first class) constraints do not influence the evaluation of the Dirac bracket of the coordinates $\{X^\mu, X^\nu\}_{\rm D}$ (in which we are interested) and, consequently, we will ignore them for our analysis. Furthermore, the analysis of the Dirac bracket using the constraints at the second boundary ($\sigma_1 = \pi$) is completely parallel to that at the first boundary ($\sigma_1 = 0$) so that we will describe the analysis using only one set of constraints, say $\varphi_{3\mu}, \varphi_{7\mu}$.

The constraints $\varphi_{3\mu}, \varphi_{7\mu}$ are second class and, therefore, the Dirac bracket between the coordinates takes the form

$$\begin{aligned}\{X_\mu(\sigma), X_\nu(\sigma')\}_{\rm D} = -\int d^2\sigma'' d^2\sigma'''\, &\{X_\mu(\sigma), \phi_A(\sigma'')\} C^{-1AB}(\sigma'', \sigma''') \times \\ &\times \{\phi_B(\sigma'''), X_\nu(\sigma')\}\,, \end{aligned} \tag{5.1}$$

where $\phi_A \equiv (\varphi_{3\mu}, \varphi_{7\mu})$ and

$$C_{AB}(\sigma, \sigma') = \begin{pmatrix} \{\varphi_{3\mu}(\sigma), \varphi_{3\nu}(\sigma')\} & \{\varphi_{3\mu}(\sigma), \varphi_{7\nu}(\sigma')\} \\ \{\varphi_{7\nu}(\sigma'), \varphi_{3\mu}(\sigma)\} & \{\varphi_{7\mu}(\sigma), \varphi_{7\nu}(\sigma')\} \end{pmatrix}. \tag{5.2}$$

We can, of course, calculate exactly all the brackets entering into the matrix, C_{AB}. However, determining the inverse matrix exactly is a technically non-trivial problem. It is here that the weak field approximation is of immense help (We want to emphasize that there is no other approximation used in our derivations.). To proceed, let us note that

$$\begin{aligned}\{\varphi_{3\mu}(\sigma), X_\lambda(\sigma')\} &= -\frac{\delta(\sigma_1)}{3}\Pi^{(U)a1}\partial_a X^\rho A_{\mu\lambda\rho}\delta(\sigma - \sigma')\,, \\ \{\varphi_{7\mu}(\sigma), X_\lambda(\sigma')\} &= \delta(\sigma_1)\left(\varepsilon^{ab} g_{b2}\partial_a\left(\frac{V}{g}\delta(\sigma - \sigma')\right)\eta_{\mu\lambda} + \right.\end{aligned}$$

$$+ \; \varepsilon^{ab}\partial_a X_\mu \partial_{(b} X_\lambda \partial_{2)} \left(\frac{V}{\bar{g}}\delta(\sigma-\sigma')\right) +$$
$$+ \; \frac{V}{3\bar{g}}\Pi^{(U)a1}A_{\mu\lambda\rho}\left((\partial_a\mathcal{P}^\rho) - \mathcal{P}^\rho\partial_a\right)\delta(\sigma-\sigma')\bigg) . \quad (5.3)$$

Here and in what follows $[a_1 \cdots a_n]$ would stand for anti-symmetrization, while $(a_1 \cdots a_n)$ would denote symmetrization of indices. We also note that, unless explicitly denoted, all quantities depend on σ (σ' dependence will be explicitly displayed). Let us parameterize these relations as

$$\begin{aligned}
\{\varphi_{3\mu}(\sigma), X_\lambda(\sigma')\} &= S_{\mu\lambda}\delta(\sigma-\sigma') , \\
\{\varphi_{7\mu}(\sigma), X_\lambda(\sigma')\} &= T_{\mu\lambda}\delta(\sigma-\sigma') + U_{\mu\lambda}\delta(\sigma-\sigma') ,
\end{aligned} \quad (5.4)$$

where

$$\begin{aligned}
S_{\mu\lambda}(\sigma) &= -\frac{\delta(\sigma_1)}{3}\Pi^{(U)a1}\partial_a X^\rho A_{\mu\lambda\rho} , \\
T_{\mu\lambda}(\sigma) &= \delta(\sigma_1)\Bigg(\varepsilon^{ab}g_{b2}\partial_a\left(\frac{V}{\bar{g}}\right)\eta_{\mu\lambda} + \\
&\qquad + \varepsilon^{ab}\partial_a X_\mu \partial_{(b} X_\lambda \partial_{2)}\left(\frac{V}{\bar{g}}\right) + \\
&\qquad + \frac{V}{\bar{g}}\left(\varepsilon^{ab}g_{b2}\eta_{\mu\lambda} + \varepsilon^{bc}\partial_b X_\mu \partial_c X_\lambda \delta^a_2 - \right. \\
&\qquad\qquad \left. - \; \varepsilon^{ab}\partial_b X_\mu \partial_2 X_\lambda\right)\partial_a\Bigg) \\
&\equiv \Omega_{\mu\lambda}(\sigma) + \Gamma^a_{\mu\lambda}(\sigma)\partial_a , \\
U_{\mu\lambda}(\sigma) &= \delta(\sigma_1)\frac{V}{3\bar{g}}\Pi^{(U)a1}A_{\mu\lambda\rho}\left((\partial_a\mathcal{P}^\rho) - \mathcal{P}^\rho\partial_a\right) .
\end{aligned} \quad (5.5)$$

Here, we have separated the Poisson bracket structures into terms that depend on $A_{\mu\nu\lambda}$ and those which do not. Thus, for example $S_{\mu\nu}$ and $U_{\mu\nu}$ are linearly dependent on $A_{\mu\nu\lambda}$ while $T_{\mu\nu}$ is independent of $A_{\mu\nu\lambda}$. This is, in fact, quite natural and useful since we are going to be expanding in powers of $A_{\mu\nu\lambda}$. Furthermore, note that it is best to think of S, T, U as operators acting on the delta function, so that S is a multiplicative operator while T and U each has a multiplicative part as well as a part that is linear in the derivative operator. For example, $\Omega_{\mu\lambda}$ represents the multiplicative operator in $T_{\mu\lambda}$, while $\Gamma^a_{\mu\lambda}\partial_a$ corresponds to the terms with the derivative operator.

Let us next analyze the structure of the inverse of the matrix C_{AB}. If we represent the matrix obtained from the mutual PB of second class constraints in the generic form as

$$C_{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad (5.6)$$

where a, b, c, d are themselves matrices (with indices suppressed for simplicity), then, it can be easily checked that the inverse has the form

$$(C^{-1})^{AB} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{5.7}$$

where

$$\begin{aligned} \alpha &= (a - bd^{-1}c)^{-1}, \\ \beta &= (c - db^{-1}a)^{-1}, \\ \gamma &= (b - ac^{-1}d)^{-1}, \\ \delta &= -b^{-1}a(c - db^{-1}a)^{-1}. \end{aligned} \tag{5.8}$$

Once we know the constraint matrix (namely, a, b, c, d), it is clear that we can calculate the inverse to any given order in $A_{\mu\nu\lambda}$ by a suitable expansion technique, using eq. (5.8). The Dirac bracket for the coordinates, of course, has the (symbolic) form

$$\{X, X\}_{\mathrm{D}} = \left(\widetilde{S}\alpha S + (\widetilde{T} + \widetilde{U})\gamma S + \widetilde{S}\beta(T + U) + (\widetilde{T} + \widetilde{U})\delta(T + U)\right), \tag{5.9}$$

where $\widetilde{S}, \ldots$ represent the transposed operators corresponding to $S, \ldots$ in the complete space.

Let us note next that we are interested in terms which are linear in $A_{\mu\nu\lambda}$ in the Dirac bracket in eq. (5.9). Since S and U are already linear in this field, the Dirac bracket simplifies and takes the form

$$\{X, X\}_{\mathrm{D}} = \left(\widetilde{T}\gamma^{(0)}S + \widetilde{S}\beta^{(0)}T + \widetilde{T}\delta^{(0)}U + \widetilde{U}\delta^{(0)}T + \widetilde{T}\delta^{(1)}T\right), \tag{5.10}$$

where, clearly, $\beta^{(0)}, \gamma^{(0)}, \delta^{(0)}$, in the first four terms on the right-hand side, do not contain $A_{\mu\nu\lambda}$, while $\delta^{(1)}$ in the last term is linear in this field (The superscript signifies the number of $A_{\mu\nu\lambda}$ contained.).

To evaluate the Dirac bracket, let us note that

$$\begin{aligned} \{\phi_{3\mu}(\sigma), \phi_{3\nu}(\sigma')\} = {} & \frac{\delta(\sigma_1)\delta(\sigma_1')}{3}\Pi^{(U)a1} \times \\ & \times \Big(-\varepsilon^{bc}\partial_a X^\lambda(\sigma')g_{2c}A_{\mu\nu\lambda}\partial_b\delta(\sigma - \sigma') + \\ & + \varepsilon^{bc}\partial_a X^\lambda g_{2c}(\sigma')A_{\mu\nu\lambda}\partial_b\delta(\sigma - \sigma') + \\ & + \varepsilon^{bc}\partial_a X^\rho(\sigma')\partial_b X_\mu A_{\nu\lambda\rho}(\partial_{(c}X^\lambda\partial_{2)}\delta(\sigma - \sigma')) - \\ & - \varepsilon^{bc}\partial_a X^\rho\partial_b X_\nu(\sigma')A_{\mu\lambda\rho}(\partial_{(c}'X^\lambda(\sigma')\partial_{2)}\delta(\sigma - \sigma')) + \\ & + \frac{3}{g}\Pi^{(U)b1}A_{\mu\lambda\rho}A_{\nu\alpha\gamma}\eta^{\alpha\rho}\mathcal{P}^\lambda\partial_a X^\gamma(\sigma')\partial_b\delta(\sigma - \sigma') - \\ & - \frac{3}{g}\Pi^{(U)b1}A_{\mu\lambda\rho}A_{\nu\alpha\gamma}\eta^{\lambda\gamma}\mathcal{P}^\alpha(\sigma')\partial_a X^\rho\partial_b'\delta(\sigma - \sigma')\Big). \end{aligned} \tag{5.11}$$

It is clear from this that the element "a" in the coefficient matrix C_{AB} is linearly and quadratically dependent on $A_{\mu\nu\lambda}$. Since we are interested only in terms which are linear in $A_{\mu\nu\lambda}$ in the Dirac bracket, we can neglect the terms quadratic in $A_{\mu\nu\lambda}$ in this Poisson bracket to write

$$\{\phi_{3\mu}(\sigma), \phi_{3\nu}(\sigma')\} = V_{\mu\nu}(\sigma,\sigma') + W_{\mu\nu}(\sigma,\sigma')\,, \tag{5.12}$$

where $V_{\mu\nu}, W_{\mu\nu}$ are, respectively the symmetric and the anti-symmetric parts (in the Lorentz index) of the terms in eq. (5.11) which are linear in $A_{\mu\nu\lambda}$, namely,

$$\begin{aligned} V_{\mu\nu}(\sigma,\sigma') = {} & \frac{\delta(\sigma_1)\delta(\sigma_1')}{6}\Pi^{(U)a1}\varepsilon^{bc} \times \\ & \times \Big(\partial_a X^\rho(\sigma')\partial_b X_{(\mu}A_{\nu)\lambda\rho}\partial_{(c}X^\lambda\partial_{2)} - \\ & \quad - \partial_a X^\rho\partial_b X_{(\nu}(\sigma')A_{\mu)\lambda\rho}\partial'_{(c}X^\lambda(\sigma')\partial_{2)}\Big)\,\delta(\sigma-\sigma') \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} W_{\mu\nu}(\sigma,\sigma') = {} & \frac{\delta(\sigma_1)\delta(\sigma_1')}{6}\Pi^{(U)a1}\varepsilon^{bc} \times \\ & \times \Big(-2\partial_a X^\lambda(\sigma')g_{2c}A_{\mu\nu\lambda}\partial_b + \\ & \quad + 2\partial_a X^\lambda g_{2c}(\sigma')A_{\mu\nu\lambda}\partial_b + \partial_a X^\rho(\sigma')\partial_b X_{[\mu}A_{\nu]\lambda\rho}\partial_{(c}X^\lambda\partial_{2)} - \\ & \quad - \partial_a X^\rho\partial_b X_{[\nu}(\sigma')A_{\mu]\lambda\rho}\partial'_{(c}X^\lambda(\sigma')\partial_{2)}\Big)\,\delta(\sigma-\sigma')\,. \end{aligned} \tag{5.14}$$

Thus, we see that since "a" is already linear in the field $A_{\mu\nu\lambda}$, it now follows from eq. (5.8) that

$$\begin{aligned} \beta^{(0)} &= c^{-1}\big|_{A_{\mu\nu\lambda}=0}\,, \\ \gamma^{(0)} &= b^{-1}\big|_{A_{\mu\nu\lambda}=0}\,, \\ \delta^{(0)} &= 0\,, \\ \delta^{(1)} &= -\,b^{-1}ac^{-1}\big|\,. \end{aligned} \tag{5.15}$$

where in the last expression b, c are restricted to be evaluated with $A_{\mu\nu\lambda} = 0$. It is worth noting from (5.15) and (5.9) that since $\delta^{(0)} = 0$, the Dirac bracket vanishes to zeroth order in the $A_{\mu\nu\lambda}$ field. This is consistent with our understanding that noncommutativity arises in the presence of an anti-symmetric background.

Since we already know the structure of the element a, it is clear that, at this order, all the relevant elements of the inverse matrix are determined completely from a knowledge of the elements b, c which are related to each other. Let us next analyze the structure of these elements. Defining,

$$M_{\mu\nu}(\sigma,\sigma') = \{\phi_{3\mu}(\sigma), \phi_{7\nu}(\sigma')\}\big|_{A_{\mu\nu\lambda}=0} \tag{5.16}$$

it is straightforward to calculate and show that

$$
\begin{aligned}
M_{\mu\nu}(\sigma,\sigma') = &-\delta(\sigma_1)\delta(\sigma_1')\varepsilon^{ab}\varepsilon^{cd}\Bigg[g_{b2}(\sigma')\partial_a'\left(\frac{V(\sigma')}{\overline{g}(\sigma')}\partial_{(d}'\delta(\sigma-\sigma')\partial_{2)}X_\nu\partial_c X_\mu\ +\right.\\
&\qquad\left.+\ \frac{V(\sigma')}{\overline{g}(\sigma')}\partial_c'\delta(\sigma-\sigma')g_{d2}\eta_{\mu\nu}\right)-\\
&\qquad -\ \partial_a X_\nu(\sigma')\partial_{(b}X_\lambda(\sigma')\partial_{2)}'\ \times\\
&\qquad \times\ \left(\frac{V(\sigma')}{\overline{g}(\sigma')}\partial_{(d}'\delta(\sigma-\sigma')\partial_{2)}X^\lambda\partial_c X_\mu\ +\right.\\
&\qquad\left.+\ \frac{V(\sigma')}{\overline{g}(\sigma')}\partial_c'\delta(\sigma-\sigma')g_{d2}\delta_\mu^\lambda\right)\Bigg]\\
= &\ G_{\mu\nu}(\sigma,\sigma')+F_{\mu\nu}(\sigma,\sigma')\,, \qquad (5.17)
\end{aligned}
$$

where $G_{\mu\nu}$ and $F_{\mu\nu}$ represent the symmetric and the antisymmetric parts of $M_{\mu\nu}$ in the Lorentz indices. More explicitly,

$$
\begin{aligned}
G_{\mu\nu}(\sigma,\sigma') = &-\frac{\delta(\sigma_1)\delta(\sigma_1')}{2}\varepsilon^{ab}\varepsilon^{cd}\Bigg[g_{d2}(\sigma')\partial_a'\left(\frac{V(\sigma')}{\overline{g}(\sigma')}\partial_{(d}'\delta(\sigma-\sigma')\partial_{2)}X_{(\nu}\partial_c X_{\mu)}\right)+\\
&+\ g_{b2}(\sigma')\partial_a'\left(\frac{V(\sigma')}{\overline{g}(\sigma')}\partial_c'\delta(\sigma-\sigma')g_{d2}\eta_{\mu\nu}\right)+\\
&+\ \partial_a X_{(\nu}(\sigma')\partial_{(2}X_\lambda(\sigma')\partial_{b)}'\ \times\\
&\times\ \left(\frac{V(\sigma')}{\overline{g}(\sigma')}\partial_{(d}'\delta(\sigma-\sigma')\partial_{2)}X^\lambda\partial_c X_{\mu)}\right)+\\
&+\ \partial_a X_{(\nu}(\sigma')\partial_{(2}X_{\mu)}(\sigma')\partial_{b)}'\left(\partial_c'\delta(\sigma-\sigma')g_{d2}\right)\Bigg],\\
F_{\mu\nu}(\sigma,\sigma') = &-\frac{\delta(\sigma_1)\delta(\sigma_1')}{2}\varepsilon^{ab}\varepsilon^{cd}\Bigg[g_{b2}(\sigma')\partial_a'\left(\frac{V(\sigma')}{\overline{g}(\sigma')}\partial_{(d}'\delta(\sigma-\sigma')\partial_{2)}X_{[\nu}\partial_c X_{\mu]}\right)+\\
&+\ \partial_a X_{[\nu}(\sigma')\partial_{(b}X_\lambda(\sigma')\partial'_{\ 2)}\ \times\\
&\times\ \left(\frac{V(\sigma')}{\overline{g}(\sigma')}\partial_{(d}'\delta(\sigma-\sigma')\partial_{2)}X^\lambda\partial_c X_{\mu]}\right)+ \qquad (5.18)\\
&+\ \partial_a X_{[\nu}(\sigma')\partial_{(b}X_{\mu]}(\sigma')\partial'_{\ 2)}\left(\frac{V(\sigma')}{\overline{g}(\sigma')}\partial_c'\delta(\sigma-\sigma')g_{d2}\right)\Bigg].
\end{aligned}
$$

Thus, we can write the inverse of $M_{\mu\nu}$ formally as

$$(M^{-1})^{\mu\nu}(\sigma,\sigma') = \left((G+F)^{-1}\right)^{\mu\nu}(\sigma,\sigma')\,. \qquad (5.19)$$

The important thing to note here is that, since neither $G_{\mu\nu}$ nor $F_{\mu\nu}$ vanishes, the inverse exists and contains both symmetric and antisymmetric parts. This, therefore, determines b^{-1} which is related to c^{-1} through a negative sign. Therefore, we are now ready to write down the Dirac bracket between the coordinates in the linear approximation in the $A_{\mu\nu\lambda}$ field.

JHEP04(2001)016

Substituting all of this into eq. (5.10), we find that the Dirac bracket has the form

$$
\begin{aligned}
\{X_\mu(\sigma), X_\nu(\sigma')\}_{\mathrm{D}} = & \Big[\widetilde{T}_{\mu\lambda}\left((G+F)^{-1}\right)^{\lambda\rho} S_{\rho\nu} - \\
& - \widetilde{S}_{\mu\lambda}\left(\left((G+F)^{-1}\right)^{\lambda\rho} T_{\rho\nu}\right) + \\
& + \widetilde{T}_{\mu\lambda}\left(\left((F+G)^{-1}(V+W)(F+G)^{-1}\right)^{\lambda\rho} T_{\rho\nu}\right)\Big](\sigma,\sigma'),
\end{aligned}
\tag{5.20}
$$

where the derivative operators in the factor T on the right are assumed to act on the factor to the left within the parenthesis with a negative sign (We will give the explicit form of this Dirac bracket in appendix B). It is worth emphasizing here that because of the structure of the boundary constraints in eqs. (4.8) and (4.9), the σ_1 coordinate is fixed at the boundary (to be $0,\pi$) and, therefore, the Dirac bracket, evaluated at equal τ, effectively depends only on the world volume coordinates τ, σ_2. This shows that the boundary string coordinates indeed become non commutative in the presence of an anti-symmetric background field and what is even more interesting is that they have a structure that is quite analogous to that in the case of strings.

6. Summary and discussions

We have studied an open membrane, with cylindrical geometry, ending on p-branes. The boundary of the open membrane on the brane is a closed string. We have confined our attention to a background configuration where the target space metric and the three-form potential are constant. We have also incorporated a two-form potential, on the boundary, whose presence is necessary to maintain gauge invariance.

The world volume action is conventionally chosen to be of Nambu-Goto form. We have adopted a modified action which has some distinct advantages as discussed in the text. We have treated the boundary conditions as primary constraints and have shown that, one can carry out the Dirac formalism without restricting to the linearized approximation of the action. We have also introduced a gauge choice, different from the one adopted in ref [5], and have shown that the Dirac procedure, in this gauge, leads to a finite number constraints. As a consequence, we are able to compute the PB brackets of all the second class constraints which are necessary for the evaluation of Dirac brackets.

Since the second class constraints are highly non linear, one, however, has to adopt an approximation scheme in order to determine the inverse of the matrix of constraints. In the absence of a length scale (i.e. there is no analogue of α' here), we evaluate the inverse matrix as well as the Dirac brackets order by order in $A_{\mu\nu\lambda}$, assuming that this constant background can be taken to be small (we have chosen background metric to be $\eta_{\mu\nu}$ for simplicity). As a consistency check, we find that the DB between coordinates on the boundary (circle) vanishes when the A-field is set to zero. We have computed the noncommutativity (strictly speaking DB) up to quadratic terms in the three form field. Since, we carry out the computations in a different gauge, our DB relations differ from ref [5] (As is well known, the noncommutativity in the Dirac brackets is gauge dependent).

We would like to note here that we have also carried out the double-dimensional reduction [16] (although we do not discuss it here) as another consistency check, where one of the coordinates say X^{10} is compactified on a circle and it is chosen to be identical to one of the world volume coordinates, say σ_2. Furthermore, as is customary for double-dimensional reduction on a circle, if we assume that the rest of the coordinates are independent of σ_2, then we obtain the reduced action corresponding to that of Cederwall and Townsend [17] as was employed in [7]. Thus, if one had started from the action [17], one could have carried out the constraint analysis through a slightly different route for open strings ending on D-branes.

We may recall, that the two form potential, introduced on the boundary to ensure gauge invariance, is actually space-time dependent for a constant field strength ($H = dB$ in form notation). Recently, it has been pointed out, in the context of strings ending on D-branes, that such space-time dependent (even for constant H-field) theories result in non-commutative field theories which do not respect associativity [18]. The $\star$ product, in such a case, is replaced by a generalized product constructed by Kontsevich [19]. Note that the boundary of the membrane is a closed string, in contrast to the case where the boundary of an open string ending on a D-brane is a point, a D0-brane. When this D0-brane couples to a gauge potential, whose field strength is taken to be constant for a constant B-field, the gauge theory becomes non commutative. As is argued in [18], for a space-time dependent B-field, in the case of open strings ending on a D-brane, on the other hand, one has a non-commutative and non-associative $\bullet$ product. In our study of open membranes with a cylindrical topology, we have closed strings on the boundary. With a space-time dependent B-field whose field strength is space-time dependent, therefore, we speculate that the underlying field theory of strings will be described by an underlying geometry which is non commutative as well as non associative. At this stage, we do not have any insight to comment on how noncommutativity of string coordinates, in general, will modify the formulation of open string field theory [20] or closed string field theory [21]. However, based on the developments in non-commutative gauge theories from the point of view of D-branes, our conjecture may have interesting implications for string field theories.

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A. Constraints in the conventional gauge choice

In this appendix, we indicate briefly how the infinite chain of constraints arises with the conventional choice of static gauge. Let us choose

$$X^0 = \tau\,, \qquad g_{0a} = 0\,, \qquad a = 1, 2\,. \tag{A.1}$$

In such a gauge, all the reparameterization invariance is fixed and the BLT action takes the form

$$S = \int d^3\xi\, L = \int d^3\xi\, \frac{1}{2V}\left(\overline{g}(1 + \dot{X}^M \dot{X}_M) - \widetilde{\mathcal{F}}^2\right). \tag{A.2}$$

Here, the indices M take only spatial values $1, 2, \ldots, 10$. The momenta, in this gauge, have the forms

$$\begin{aligned}
P_M &= \frac{\partial L}{\partial \dot{X}^M} = \frac{\overline{g}}{V}\dot{X}_M - \frac{1}{2V}\widetilde{\mathcal{F}}\varepsilon^{ab} A_{MNP}\partial_a X^N \partial_b X^P\,,\\
\Pi^{(U)ab} &= \frac{\partial L}{\partial \dot{U}_{ab}} = -\frac{3\varepsilon^{ab}}{V}\widetilde{\mathcal{F}}\,,\\
\Pi^{(U)0a} &= \frac{\partial L}{\partial \dot{U}_{0a}} \approx 0\,,\\
P_V &= \frac{\partial L}{\partial \dot{V}} \approx 0\,.
\end{aligned} \tag{A.3}$$

Thus, we see that the velocities can be inverted and that we have two primary constraints as before. Let us define, as in the analysis of section 3 (except that now indices take only spatial values),

$$\mathcal{P}_M = P_M - \frac{1}{6}\,\Pi^{(U)ab} A_{MNP}\partial_a X^N \partial_b X^P\,. \tag{A.4}$$

With this, as well as our assumption of a cylindrical topology for the open membrane, the boundary conditions take the form

$$\left[\left(\mathcal{P}^2 + \left(\frac{\overline{g}}{V}\right)^2\right)\overline{g}^{a1}\partial_a X_\mu + \frac{1}{3}\Pi^{(U)a1} A_{\mu\nu\lambda}\mathcal{P}^\nu \partial_a X^\lambda\right]_{\sigma_1 = 0,\pi} \approx 0\,. \tag{A.5}$$

Thus, we can write all the primary constraints, in this gauge, to be

$$\begin{aligned}
\varphi_1 &= P_V \approx 0\,,\\
\varphi_2^a &= \Pi^{(U)0a} \approx 0\,,\\
\varphi_{3\mu} &= \left(\left(\mathcal{P}^2 + \left(\frac{\overline{g}}{V}\right)^2\right)\overline{g}^{a1}\partial_a X_\mu + \frac{1}{3}\Pi^{(U)a1} A_{\mu\nu\lambda}\mathcal{P}^\nu \partial_a X^\lambda\right)\delta(\sigma_1) \approx 0\,,\\
\varphi_{4\mu} &= \left(\left(\mathcal{P}^2 + \left(\frac{\overline{g}}{V}\right)^2\right)\overline{g}^{a1}\partial_a X_\mu + \frac{1}{3}\Pi^{(U)a1} A_{\mu\nu\lambda}\mathcal{P}^{nu} \partial_a X^\lambda\right)\delta(\sigma_1 - \pi) \approx 0\,.
\end{aligned} \tag{A.6}$$

The total hamiltonian for the system, including the primary constraints, has the form,

$$\begin{aligned} H = {} & \frac{V}{2\overline{g}}\left(\mathcal{P}^2 - \left(\frac{\overline{g}}{V}\right)^2\right) - \frac{V}{72}\left(\varepsilon_{ab}\Pi^{(U)ab}\right)^2 + 2\Pi^{(U)ab}\partial_a U_{0b} + \\ & + c\varphi_1 + k_a\varphi_2^a + \lambda^\mu\varphi_{3\mu} + \widetilde{\lambda}^\mu\varphi_{4\mu}\,. \end{aligned} \tag{A.7}$$

So far everything seems completely parallel to the discussion in section 4 except for the extra $(\overline{g}/V)^2$ terms in the boundary conditions as well as in the hamiltonian. As we will see, these make all the difference.

The constraint analysis can be carried out now. Requiring $\dot{\varphi}_1 \approx 0$ leads to

$$\begin{aligned} \lambda^\mu &= 0 = \widetilde{\lambda}^\mu\,, \\ \varphi_5 &= \frac{1}{2\overline{g}}\left(\mathcal{P}^2 + \left(\frac{\overline{g}}{V}\right)^2\right) - \frac{1}{72}\left(\varepsilon_{ab}\Pi^{(U)ab}\right)^2 \approx 0\,. \end{aligned} \tag{A.8}$$

As before, there is a secondary constraint, which, however, does not correspond to the vanishing of the hamiltonian. In fact, there is no reason for it to, since we have already gauge fixed the τ reparameterization invariance. Requiring $\dot{\varphi}_5 \approx 0$, as is easily seen, determines the Lagrange multiplier c. This is already a point of departure from the earlier analysis. (Namely, earlier, the secondary condition implied the vanishing of the hamiltonian which is automatically invariant under time evolution. But in the present case, the new constraint does not correspond to the vanishing of the hamiltonian and, therefore, leads to a non-trivial relation, namely it determines the Lagrange multiplier c.)

Requiring $\dot{\varphi}_2^a \approx 0$ leads, as before, to the Gauss' Law constraint

$$\varphi_6^a = \partial_b \Pi^{(U)ab} \approx 0 \tag{A.9}$$

which does not generate any further constraint. Turning now to the boundary constraints, we recognize that, since the Lagrange multiplier c is already determined and that the boundary constraints do not depend on the field U_{0a}, their consistency can only lead to new (secondary) constraints which, in turn, will lead to tertiary constraints and so on. Explicit calculation, indeed, do verify this. Namely, in this gauge, the boundary constraints do lead to an infinite chain of constraints as is normally expected.

B. Dirac brackets up to second order in $A_{\mu\nu\lambda}$

In this appendix, we will give the explicit form of the Dirac bracket, linear in $A_{\mu\nu\lambda}$, defined in eq. (5.20) as well as indicate the structure of the Dirac bracket up to quadratic terms in $A_{\mu\nu\lambda}$.

JHEP04(2001)016

Let us recall that (see eq. (5.15))

$$\begin{aligned}
\left(\beta^{(0)}\right)^{\mu\nu} &= \left(c^{-1}\right)^{\mu\nu}\big|_{A_{\mu\nu\lambda}=0} = -\left((F+G)^{-1}\right)^{\mu\nu}, \\
\left(\gamma^{0}\right)^{\mu\nu} &= \left(b^{-1}\right)^{\mu\nu}\big|_{A_{\mu\nu\lambda}=0} = \left((F+G)^{-1}\right)^{\mu\nu}, \\
\left(\delta^{(1)}\right)^{\mu\nu} &= -\left(b^{-1}ac^{-1}\right)^{\mu\nu}\big| = \left((F+G)^{-1}(V+W)(F+G)^{-1}\right)^{\mu\nu}.
\end{aligned} \tag{B.1}$$

The Dirac bracket between the coordinates, linear in $A_{\mu\nu\lambda}$ (see eq. (5.20)), can now be written out explicitly as

$$\begin{aligned}
&\{X_\mu(\sigma), X_\nu(\sigma')\}_{\mathrm{D}} = \\
&\quad = \widetilde{\Omega}_{\mu\lambda}(\sigma)\left(\gamma^{(0)}\right)^{\lambda\rho}(\sigma,\sigma')S_{\rho\nu}(\sigma') - \partial_a\left(\widetilde{\Gamma}^a_{\mu\lambda}(\sigma)\left(\gamma^{(0)}\right)^{\lambda\rho}(\sigma,\sigma')S_{\rho\nu}(\sigma')\right) - \\
&\qquad - \widetilde{S}_{\mu\lambda}(\sigma)\left(\beta^{(0)}\right)^{\lambda\rho}(\sigma,\sigma')\Omega_{\rho\nu}(\sigma') + \partial'_a\left(\widetilde{S}_{\mu\lambda}(\sigma)\left(\beta^{(0)}\right)^{\lambda\rho}(\sigma,\sigma')\Gamma_{\rho\nu}(\sigma')\right) + \\
&\qquad + \widetilde{\Omega}_{\mu\lambda}(\sigma)\left(\delta^{(1)}\right)^{\lambda\rho}(\sigma,\sigma')\Omega_{\rho\nu}(\sigma') - \partial_a\left(\widetilde{\Gamma}^a_{\mu\lambda}(\sigma)\left(\delta^{(1)}\right)^{\lambda\rho}(\sigma,\sigma')\Omega_{\rho\nu}(\sigma')\right) + \\
&\qquad - \partial'_a\left(\widetilde{\Omega}_{\mu\lambda}(\sigma)\left(\delta^{(1)}\right)^{\lambda\rho}(\sigma,\sigma')\Gamma^a_{\rho\nu}(\sigma')\right) + \\
&\qquad + \partial_a\partial'_b\left(\widetilde{\Gamma}^a_{\mu\lambda}(\sigma)\left(\delta^{(1)}\right)^{\lambda\rho}(\sigma,\sigma')\Gamma^b_{\rho\nu}(\sigma')\right).
\end{aligned} \tag{B.2}$$

We note here that the Dirac bracket can be specified completely in terms of the coordinates and the $A_{\mu\nu\lambda}$ field with the identification (see eqs. (4.4) and (4.7))

$$\Pi^{(U)ab} = 3\varepsilon^{ab}. \tag{B.3}$$

Let us next indicate briefly the structure of the Dirac bracket between coordinates up to quadratic order in the $A_{\mu\nu\lambda}$ fields. From eq. (5.9) and from the structure of various quantities in there, we see that up to quadratic order in $A_{\mu\nu\lambda}$, the Dirac bracket will contain, in addition to the terms on the right-hand side in eq. (B.2), terms which are quadratic in $A_{\mu\nu\lambda}$. These can be written symbolically as

$$\begin{aligned}
\{X_\mu, X_\nu\}^{(2)}_{\mathrm{DB}} = &\left(\widetilde{S}\alpha^{(0)}S + \widetilde{U}\gamma^{(0)}S + \widetilde{S}\beta^{(0)}U\right) + \\
&+ \left(\widetilde{T}\gamma^{(1)}S + \widetilde{S}\beta^{(1)}T + \widetilde{T}\delta^{(1)}U + \widetilde{U}\delta^{(1)}T\right) + \widetilde{T}\delta^{(2)}T.
\end{aligned} \tag{B.4}$$

To determine this, let us note that we can decompose and write the elements of the matrix of constraints as a series of terms containing different powers of $A_{\mu\nu\lambda}$. Namely, let us write

$$\begin{aligned}
a &= a^{(1)} + a^{(2)}, \\
b &= b^{(0)} + b^{(1)} + b^{(2)}, \\
c &= c^{(0)} + c^{(1)} + c^{(2)}, \\
d &= d^{(0)} + d^{(1)} + d^{(2)}.
\end{aligned} \tag{B.5}$$

It is important to recognize that, since the constraints are at best linear in $A_{\mu\nu\lambda}$, the elements of the constraint matrix can at most have quadratic dependence on $A_{\mu\nu\lambda}$.

With this, we can determine the elements of the inverse matrix perturbatively. At the zeroth order, they take the forms

$$
\begin{aligned}
\alpha^{(0)} &= -\left(c^{(0)}\right)^{-1} d^{(0)} \left(b^{(0)}\right)^{-1}, \\
\beta^{(0)} &= \left(c^{(0)}\right)^{-1}, \\
\gamma^{(0)} &= \left(b^{(0)}\right)^{-1}, \\
\delta^{(0)} &= 0\,.
\end{aligned} \tag{B.6}
$$

At linear order, they are determined to be

$$
\begin{aligned}
\alpha^{(1)} &= -\alpha^{(0)} \Big(a^{(1)} - b^{(0)} \left(d^{(0)}\right)^{-1} c^{(1)} - b^{(1)} \left(d^{(0)}\right)^{-1} c^{(0)} + \\
&\qquad\qquad + b^{(0)} \left(d^{(0)}\right)^{-1} d^{(1)} \left(d^{(0)}\right)^{-1} c^{(0)}\Big)\, \alpha^{(0)}, \\
\beta^{(1)} &= -\beta^{(0)} \Big(c^{(1)} - d^{(0)} \left(b^{(0)}\right)^{-1} a^{(1)}\Big)\, \beta^{(0)}, \\
\gamma^{(1)} &= -\gamma^{(0)} \Big(b^{(1)} - a^{(1)} \left(c^{(0)}\right)^{-1} d^{(0)}\Big)\, \gamma^{(0)}, \\
\delta^{(1)} &= -\left(b^{(0)}\right)^{-1} a^{(1)} \left(c^{(0)}\right)^{-1}.
\end{aligned} \tag{B.7}
$$

And, finally, the last term that is needed for the Dirac bracket at the quadratic order (B.4) is determined to be

$$
\begin{aligned}
\delta^{(2)} = {} & \left(b^{(0)}\right)^{-1} a^{(1)} \left(c^{(0)}\right)^{-1} \Big(c^{(1)} - d^{(0)} \left(b^{(0)}\right)^{-1} a^{(1)}\Big) \left(c^{(0)}\right)^{-1} - \\
& - \left(b^{(0)}\right)^{-1} \Big(a^{(2)} - b^{(1)} \left(b^{(0)}\right)^{-1} a^{(1)}\Big) \left(c^{(0)}\right)^{-1}.
\end{aligned} \tag{B.8}
$$

Substituting these into eq. (B.4) determines explicitly the Dirac bracket for the coordinates that is quadratic in the $A_{\mu\nu\lambda}$ fields. In fact, this iterative procedure can be carried out to any order in the field $A_{\mu\nu\lambda}$

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